Available online at www.sciencedirect.com

SCIENCE DIRECT®

Discrete Mathematics 294 (2005) 21–24

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

# On quasi-3 designs and spin models

Carl Bracken, Gary McGuire

*Department of Mathematics, National University of Ireland (NUI), Maynooth, Co. Kildare, Ireland*

Received 16 April 2003; received in revised form 18 June 2003; accepted 29 April 2004
Available online 4 March 2005

**Abstract**

A theorem of Bannai and Sawano shows that certain four-weight spin models exist if and only if certain quasi-3 designs exist. We verify that the known quasi-3 designs, other than SDP designs, do not give rise to four-weight spin models.



*MSC:* 05B05; 57M27

*Keywords:* Spin model; Quasi-3 design; Symmetric design

## 1. Introduction

The concept of a *spin model* was introduced by Jones [9]. The concept was generalised to *two-weight* spin models by Kawagoe et al. [10], and further generalised to *four-weight* spin models by Bannai and Bannai [1].

Guo and Huang [8] considered certain types of four-weight spin models, which they called "four-weight spin models with exactly two values on $W_2$". They showed a connection with symmetric designs. Bannai and Sawano [2] showed that the existence of a four-weight spin model with exactly two values on $W_2$ is equivalent to the existence of a quasi-3 design with certain properties (see Theorem 1 below), strengthening the result of Guo and Huang [8]. In this note we investigate whether the known quasi-3 designs, apart from SDP designs, satisfy these additional properties. None of them do. In a separate article [3] we investigate SDP designs.

*E-mail address:* gmg@maths.may.ie (G. McGuire).

0012-365X/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2004.04.033

## 2. Background

We refer the reader to [2] or [1] for the definition of a four-weight spin model, and to [7] for the basic properties of block designs.

A symmetric design is said to be *quasi*-3 *for points* if the number of blocks incident with three distinct points takes only two values. Such designs seem to have been first considered in [6]; see also [5] for a survey of quasi-3 designs. We shall say that a symmetric design is *quasi*-3 *for blocks* if the number of points in the intersection of any three distinct blocks takes only two values. A design is quasi-3 for blocks if and only if the dual design is quasi-3 for points.

In [5] Broughton and the second author have surveyed the known (at that time) quasi-3 designs. These fall into the following categories (up to complementation):

- Projective geometries $\mathrm{PG}(d, q)$.
- SDP designs.
- Biplanes (square designs with $\lambda = 2$).
- Kronecker products of quasi-3 $(4u^2, 2u^2 - u, u^2 - u)$ designs.

We now state the result of [2].

**Theorem 1** (*Bannai and Sawano [2]*). *Let $W_2 = \alpha A + \beta(J - A)$, where $\alpha, \beta$ are distinct nonzero complex numbers, and A is a $(0, 1)$-matrix with the property that each row and column has exactly k ones, where $2 \leqslant k \leqslant n - 2$. Let X be a finite set with n elements, and let D be a real number satisfying $D^2 = n$. Then $W_2$ defines a four-weight spin model if and only if A is the incidence matrix of a symmetric $(n, k, \lambda)$ design $\mathscr{D}(X, \mathscr{B})$ which satisfies the following three properties*:

(1) *$\mathscr{D}(X, \mathscr{B})$ has only two triple intersection sizes for blocks, which are*

$$x = \frac{k\lambda - k + \lambda - (k - \lambda)\sqrt{k - \lambda}}{n}, \quad y = \frac{k\lambda - k + \lambda + (k - \lambda)\sqrt{k - \lambda}}{n}.$$

(2) *For any set $\mathscr{S} \subseteq \mathscr{B}$ of four blocks, an even number of the four 3-subsets of $\mathscr{S}$ have triple intersection size x.*

(3) *There exists a 1–1 correspondence $\phi : X \longrightarrow \mathscr{B}$ with the property that for any three points $a, b, c \in X$, the number of blocks containing $\{a, b, c\}$ is $|\phi(a) \cap \phi(b) \cap \phi(c)|$.*

*Moreover, if conditions (1)–(3) hold, then $\alpha, \beta$ and $W_1$ are determined by D and k. In particular, $\alpha = -\beta$ if and only if $n = 4q^2$, where q is an even integer.*

It is clear that a design satisfying condition 1 is quasi-3 for blocks.

It was pointed out to us by several people at the Irsee conference that condition 2 is equivalent to $\mathscr{B}$ being the point set of a regular two-graph (see [7] for the definition of regular two-graph). A regular two-graph yields a strongly regular graph. It is well known that a quasi-symmetric design also yields a strongly regular graph. Thus, a design satisfying conditions (1)–(3) of Theorem 1 will give rise to possibly two strongly regular graphs.

However, it is not hard to see from the constructions (which can both be found in [7]) that these graphs are the same.

Condition (3) is certain to hold if the design has a polarity. We do not know if the converse is true.

Guo and Huang [8] point out that the $(16, 6, 2)$ SDP design satisfies conditions (1)–(3) of Theorem 1, and thus gives an example of a four-weight spin model with exactly two values on $W_2$. Bannai and Sawano [2] showed that the other (non-SDP) $(16, 6, 2)$ designs do not satisfy conditions (1)–(3) of Theorem 1.

In this note we will verify that the known classes of quasi-3 designs listed above (excluding SDP designs) do not satisfy all of conditions (1)–(3) of Theorem 1. In a separate article [3] we investigate SDP designs.

## 3. Verification

In this section we will check that the known classes of quasi-3 designs in [5] (apart from SDP designs) do not give rise to spin models via Theorem 1.

The classes we must check are

- Projective geometries $\mathrm{PG}(d, q)$.
- Biplanes (square designs with $\lambda = 2$).
- Kronecker products of quasi-3 $(4u^2, 2u^2 - u, u^2 - u)$ designs.

1. *Projective geometries* $\mathrm{PG}(d, q)$: The point-hyperplane design $\mathrm{PG}(d, q)$ is quasi-3 with $y = \lambda$. Using the expression for $y$ in Theorem 1 we obtain

$$k^2 - k + \lambda = k\lambda - k + \lambda + (k - \lambda)\sqrt{k - \lambda},$$

which when rearranged gives $k^2 - k + \lambda = 0$. This implies $k = 1$ and $\lambda = 0$, which is a trivial case. Thus $\mathrm{PG}(d, q)$ does not satisfy condition 1 of Theorem 1.

2. *Biplanes*: Since $\lambda = 2$ we must have $x = 0$ and $y = 1$. Using the expression for $x$ in Theorem 1 we obtain

$$2k - k + 2 - (k - 2)\sqrt{k - 2} = 0.$$

This gives $(k + 2)^2 = (k - 2)^3$, yielding a cubic polynomial for $k$. The only integer solution is $k = 6$, and it follows that the only biplanes which can possibly satisfy the conditions of Theorem 1 have parameters $(16, 6, 2)$. As we mentioned in the previous section, only the SDP $(16, 6, 2)$ design with those parameters satisfies all three conditions.

3. *Kronecker products*: Let $H$ be a quasi-3 $(4u^2, 2u^2 - u, u^2 - u)$ design, and let $K$ be a quasi-3 $(4w^2, 2w^2 - w, w^2 - w)$ design. Then $M = H \otimes K$ is a quasi-3 $(4(2uw)^2, 2(2uw)^2 - 2uw, (2uw)^2 - 2uw)$ design, as shown in [4]. The triple intersection sizes in $H$ are $x_H = u(u - 2)/2$, and $y_H = u(u - 1)/2$, with corresponding values for $K$ and $M$.

Suppose that $H$ does not satisfy condition 2 of Theorem 1. We will show that $M$ does not satisfy condition 2 either.

Let $F$ be the $4 \times 4u^2$ matrix representing four blocks in $H$ that do not obey condition 2, i.e., an odd number of triples of these blocks have triple intersection size $x_H$.

Then, up to a permutation of rows and columns, there are four blocks in $M$ that can be represented by the $4 \times 4(2uw)^2$ matrix

$$\overline{F}\overline{F} \ldots \overline{F}\overline{F} F F \ldots F F,$$

where there are $2w^2 - w$ $\overline{F}$'s, and $2w^2 + w$ $F$'s.

If three of these four blocks are chosen corresponding to three blocks in $H$ having triple intersection size $x_H$, then the triple intersection size will be

$$(2w^2 - w)(u^2 - x_H) + (2w^2 + w)(x_H),$$

which is equal to $x_M$. Similarly, the triple intersection size $y_H$ in $H$ yields a triple intersection size $y_M$ in $M$.

It is of course possible that a design satisfies conditions 1 and 2, but not condition 3, although no examples of this are known. Condition 2 seems the most stringent. We have so far only checked condition 2 for the Kronecker product class, and not condition 3. To finish, we briefly discuss condition 3.

If $H$ and $K$ both satisfy condition 3, then $H \otimes K$ will also satisfy condition 3. The proof of this is clear: a choice of three points in $H \otimes K$ is equivalent to a choice of three points in $H$ and a choice of three points in $K$, and the property follows.

However, we have found an example of two designs, neither of which satisfies condition 3, but whose Kronecker product does. The example is $D$, $D^{\mathrm{T}}$, and $D \otimes D^{\mathrm{T}}$, where $D$ is a quasi-3 (64, 28, 12) design whose dual design $D^{\mathrm{T}}$ is quasi-3 but not isomorphic to $D$. Since $D \otimes D^{\mathrm{T}}$ has a polarity it must satisfy condition 3, but we have checked that neither $D$ or $D^{\mathrm{T}}$ do. All of these designs obey condition 1, but none obey condition 2.

## References


[1] E. Bannai, E. Bannai, Generalized spin models (four-weight spin models), Pacific J. Math. 170 (1995) 1–16.
[2] E. Bannai, M. Sawano, Symmetric designs attached to four-weight spin models, Des. Codes Cryptogr. 25 (2002) 73–90.
[3] C. Bracken, G. McGuire, Characterization of SDP designs that yield certain spin models, Des. Codes Cryptogr., to appear.
[4] W. Broughton, G. McGuire, Some observations on quasi-3 designs and Hadamard matrices, Des. Codes Cryptogr. 18 (1999) 55–61.
[5] W. Broughton, G. McGuire, On the non-existence of quasi-3 designs, Discrete Math. 262 (2003) 79–87.
[6] P.J. Cameron, Near-regularity conditions for designs, Geom. Dedicata 2 (1973) 213–223.
[7] P.J. Cameron, J.H. van Lint, Designs, Graphs, Codes and their Links, Cambridge University Press, Cambridge, 1991.
[8] H. Guo, T. Huang, Some classes of four-weight spin models, J. Statist. Plann. Inference 94 (2001) 231–247.
[9] V.F.R. Jones, On knot invariants related to some statistical mechanical models, Pacific J. Math. 137 (1989) 311–334.
[10] K. Kawagoe, A. Munemasa, Y. Watatani, Generalised spin models, J. Knot Theory Ramif. 4 (1994) 465–475.